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Unsteady viscous flow over a wavy wall

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The method of conformal transformation is used to investigate the steady streaming generated by an oscillatory viscous flow over a wavy wall. By assuming that the amplitude of the wall is much smaller than the Stokes layer thickness, the equations are linearized and solved for large and small values of the parameter kR. This parameter is the ratio of the amplitude of oscillation of a fluid particle to the wavelength of the wall. When $kR \ll 1$, the results due to Schlichting (1932) are recovered, and when $kR \gg 1$ the equations resemble closely those derived in the theory of stability of plane parallel flows. With the aid of this theory the first-order steady streaming is found.

1. Introduction

When a purely oscillatory viscous flow is set up over a curved surface the Reynolds stresses within the fluid generate a steady streaming. This has attracted the attention of several authors, notably Schlichting (1932), Riley (1965) and Stuart (1966). Although they principally considered flow over a cylinder, their conclusions are in general true for any two-dimensional surface. Their analyses all make use of the assumption that the amplitude of oscillation of a fluid particle far from the surface is much smaller than a typical dimension of that surface.

In order to study the effect of allowing the amplitude of the oscillation to be much greater than such a dimension, we assume here that the surface is a small perturbation to an infinite flat plate. This enables us to use the method of conformal transformations to solve the viscous flow equations, a method employed by Segel (1961).

We recover the results due to Schlichting for small amplitudes of oscillation, but also discover that the problem at large amplitudes is tractable. In fact we find that the theory is almost identical to that for the stability of plane parallel flows, and extensive use is made of existing knowledge of the latter to solve for the steady streaming. The difference lies in the order of magnitude of the time derivative, and this enables us to treat the time variable solely as a parameter. This important simplification allows us to follow closely the analysis of Benjamin (1959) who treated the steady problem.

2. Formulation of the problem

Let us consider two-dimensional viscous flow over an infinite wall, the surface of which is defined by

$$y = \alpha \cos \kappa x, \tag{2.1}$$

where x, y are rectangular Cartesian co-ordinates. Thus the wall has a wavelength $2\pi/\kappa$ and an amplitude α. Writing $z = x+iy$, we consider the following conformal transformation

$$\zeta = \Psi + i\Phi = z - i\alpha\, e^{i\kappa z}. \tag{2.2}$$

If α is small, the Jacobian J of this transformation is

$$J = |d\zeta/dz|^2 = 1 + 2\alpha\kappa\, e^{-\kappa y}\cos\kappa x + O(\alpha^2) \tag{2.3}$$

and, equating real and imaginary parts, we have

$$\left.\begin{aligned}\Psi &= x + \alpha\, e^{-\kappa y}\sin\kappa x,\\ \Phi &= y - \alpha\, e^{-\kappa y}\cos\kappa x.\end{aligned}\right\} \tag{2.4}$$

Hence we obtain

$$J = 1 + 2\alpha\kappa\, e^{-\kappa\Phi}\cos\kappa\Psi + O(\alpha^2). \tag{2.5}$$

The surface of the wall is now defined in these transformed co-ordinates by $\Phi = O(\alpha^2)$ and, because we shall be neglecting terms of $O(\alpha^2)$ in the following analysis, this may be replaced by $\Phi = 0$.

Let $\mathbf{u} = (u, v)$ be the velocity vector in the transformed co-ordinate system. Thus u is the component of velocity in the Ψ increasing direction, and v is the component in the Φ increasing direction. In addition let p denote pressure, ρ the density of the fluid and ν its kinematic viscosity. The momentum equation for the flow is then

$$\partial\mathbf{u}/\partial t + \operatorname{grad}(\tfrac{1}{2}\mathbf{u}^2) - \mathbf{u}\times\operatorname{curl}\mathbf{u} = -(1/\rho)\operatorname{grad}p - \nu\operatorname{curl}\operatorname{curl}\mathbf{u} \tag{2.6}$$

and the equation of continuity is

$$\operatorname{div}\mathbf{u} = 0. \tag{2.7}$$

Because the transformation is conformal, the line element ds in the transformed co-ordinates is

$$ds^2 = J^{-1}(d\Psi^2 + d\Phi^2) \tag{2.8}$$

and hence (2.7) becomes

$$J\left[\frac{\partial}{\partial\Psi}(J^{-\frac{1}{2}}u) + \frac{\partial}{\partial\Phi}J(^{-\frac{1}{2}}v)\right] = 0. \tag{2.9}$$

In order to satisfy (2.9) we define a stream function X such that

$$u = J^{\frac{1}{2}}\,\partial \mathrm{X}/\partial\Phi, \quad v = -J^{\frac{1}{2}}\,\partial \mathrm{X}/\partial\Psi \tag{2.10}$$

and if we now eliminate the pressure p from (2.6) by taking the curl of both sides of the equation, we obtain the following equation for X.

$$\frac{\partial}{\partial t}D^2\mathrm{X} - \frac{\partial(\mathrm{X}, JD^2\mathrm{X})}{\partial(\Psi,\Phi)} = \nu D^2(JD^2\mathrm{X}), \tag{2.11}$$

where

$$D^2 \equiv \partial^2/\partial\Psi^2 + \partial^2/\partial\Phi^2. \tag{2.12}$$

The boundary conditions we wish to impose on the problem are the conditions of zero slip on the wall and, as $y\to\infty$, the value of the velocity vector should be $U_\infty\cos\omega t$ in the x direction. In the transformed co-ordinate system these conditions become

$$\left.\begin{aligned}&\mathrm{X} = \partial\mathrm{X}/\partial\Phi = 0 \quad \text{on} \quad \Phi = 0,\\ &\left.\begin{aligned}\partial\mathrm{X}/\partial\Phi &\to U_\infty\cos\omega t\\ \partial\mathrm{X}/\partial\Psi &\to 0\end{aligned}\right\} \quad \text{as} \quad \Phi\to\infty.\end{aligned}\right\} \tag{2.13}$$

In addition only harmonic dependence on ωt will be allowed.

If the wall were flat, i.e. $J = 1$, then the solution of (2·11) subject to (2·13) would be the well-known Stokes shear-wave solution

$$X = U_\infty\{\Phi \cos \omega t + (\nu/\omega)^{\frac{1}{2}} [e^{-(\omega/2\nu)^{\frac{1}{2}}\Phi} \sin (\omega t - (\omega/2\nu)^{\frac{1}{2}} \Phi + \tfrac{1}{4}\pi) - \sin (\omega t + \tfrac{1}{4}\pi)]\}. \tag{2.14}$$

We consider here the case where the amplitude of the wave α is finite but also $\alpha \ll O(2\nu/\omega)^{\frac{1}{2}}$. In other words the amplitude of the wave is much smaller than the thickness of the Stokes layer, and we may therefore expect (2.14) to be a first approximation to the solution. Since the Stokes layer has a characteristic dimension of $O(2\nu/\omega)^{\frac{1}{2}}$, we may define the following non-dimensional notation:

$$\left.\begin{aligned} &\chi = X\left[U_\infty\left(\frac{2\nu}{\omega}\right)^{\frac{1}{2}}\right]^{-1}, \quad \psi = \Psi\left(\frac{2\nu}{\omega}\right)^{-\frac{1}{2}}, \\ &\eta = \Phi\left(\frac{2\nu}{\omega}\right)^{-\frac{1}{2}}, \quad \tau = \omega t, \quad a = \alpha\left(\frac{2\nu}{\omega}\right)^{-\frac{1}{2}}, \\ &k = \kappa\left(\frac{2\nu}{\omega}\right)^{\frac{1}{2}}, \quad R = \frac{U_\infty 2^{\frac{1}{2}}}{(\nu\omega)^{\frac{1}{2}}}, \end{aligned}\right\} \tag{2.15}$$

where R is the Reynolds number.

Equation (2.11) now becomes

$$2R^{-1}\frac{\partial}{\partial \tau} D^2\chi - \frac{\partial(\chi, JD^2\chi)}{\partial(\psi, \eta)} = R^{-1}D^2(JD^2\chi), \tag{2.16}$$

where $\quad D^2 \equiv \partial^2/\partial\psi^2 + \partial^2/\partial\eta^2 \quad$ and $\quad J = 1 + 2ak\,e^{-k\eta} \cos k\psi + O(a^2).$ (2.17)

In addition the boundary conditions (2.13) become

$$\left.\begin{aligned} &\chi = \partial\chi/\partial\eta = 0 \quad \text{on} \quad \eta = 0, \\ &\left.\begin{aligned}&\partial\chi/\partial\eta \to \cos\tau \\ &\partial\chi/\partial\psi \to 0\end{aligned}\right\} \quad \text{as} \quad \eta \to \infty. \end{aligned}\right\} \tag{2.18}$$

We now look for a solution to (2.16) of the form

$$\chi = \chi_0 + a\chi_1 + a^2\chi_2 + \ldots, \tag{2.19}$$

remembering that for this problem $a \ll 1$. Substituting (2.19) into (2.16) and equating like powers of a we have, as our equation for χ_0,

$$2R^{-1}\frac{\partial}{\partial \tau} D^2\chi_0 - \frac{\partial(\chi_0, D^2\chi_0)}{\partial(\psi, \eta)} = R^{-1}D^4\chi_0, \tag{2.20}$$

whose solution must satisfy the boundary conditions

$$\left.\begin{aligned} &\chi_0 = \partial\chi_0/\partial\eta = 0 \quad \text{on} \quad \eta = 0, \\ &\left.\begin{aligned}&\partial\chi_0/\partial\eta \to \cos\tau \\ &\partial\chi_0/\partial\psi \to 0\end{aligned}\right\} \quad \text{as} \quad \eta \to \infty. \end{aligned}\right\} \tag{2.21}$$

As expected, this λ is just the Stokes shear-wave solution

$$\chi_0 = \eta \cos\tau + 2^{-\frac{1}{2}}[e^{-\eta} \sin(\tau - \eta + \tfrac{1}{4}\pi) - \sin(\tau + \tfrac{1}{4}\pi)]. \tag{2.22}$$

The equation of $O(a)$ from (2.16) is

$$2R^{-1}\frac{\partial}{\partial\tau}D^2\chi_1-\frac{\partial(\chi_0,D^2\chi_1)}{\partial(\psi,\eta)}-\frac{\partial(\chi_1,D^2\chi_0)}{\partial(\psi,\eta)}-\frac{\partial(\chi_0,2k\,e^{-k\eta}\cos k\psi D^2\chi_0)}{\partial(\psi,\eta)}$$
$$=R^{-1}\{D^4\chi_1+D^2(2k\,e^{-k\eta}\cos k\psi D^2\chi_0)\},\quad (2.23)$$

whose solution must satisfy the boundary conditions

$$\left.\begin{array}{ll}\chi_1=\partial\chi_1/\partial\eta=0 & \text{on}\quad \eta=0,\\ \left.\begin{array}{l}\partial\chi_1/\partial\eta\to 0\\ \partial\chi_1/\partial\psi\to 0\end{array}\right\} & \text{as}\quad \eta\to\infty.\end{array}\right\}\quad (2.24)$$

We define $U(\eta,\tau)=\partial\chi_0/\partial\eta$, i.e.

$$U(\eta,\tau)=\cos\tau-e^{-\eta}\cos(\tau-\eta),\quad (2.25)$$

then, following Benjamin (1959), we write

$$\chi_1=\mathscr{R}\{F(\eta,\tau)+U(\eta,\tau)\,e^{-k\eta}\,e^{ik\psi}\},\quad (2.26)$$

where $\mathscr{R}$ denotes 'real part of'.

Equation (2.23) now becomes

$$\frac{2}{ikR}\frac{\partial}{\partial\tau}(F''-k^2F)+U(F''-k^2F)-U''F=\frac{1}{ikR}(F^{\text{iv}}-2k^2F''+k^4F),\quad (2.27)$$

with boundary conditions

$$\left.\begin{array}{ll}F=0,\,F'=-U' & \text{on}\quad \eta=0,\\ \left.\begin{array}{l}F'\to 0\\ F\to 0\end{array}\right\} & \text{as}\quad \eta\to\infty,\end{array}\right\}\quad (2.28)$$

where a prime denotes differentiation with respect to η.

Equation (2.28) is almost identical to the Orr–Sommerfeld equation which arises in the theory of stability of plane parallel flows, and when the parameter kR is large we shall make use of this theory in solving the equation. This parameter is clearly of considerable importance in deriving the steady streaming, and we may observe that $kR/4\pi$ is, in fact, the ratio of the amplitude of oscillation of the fluid far from the wall (U_∞/ω) to the wavelength of the wall $(2\pi/\kappa)$.

If we write the non-dimensional velocity u in the ψ direction as

$$u=u_0+au_1+a^2u_2+\ldots,\quad (2.29)$$

then, as a consequence of (2.10), (2.15) and (2.17), we find that

$$u_0=\frac{\partial\chi_0}{\partial\eta},\quad u_1=k\,e^{-k\eta}\cos k\psi\frac{\partial\chi_0}{\partial\eta}+\frac{\partial\chi_1}{\partial\eta}.\quad (2.30)$$

Hence, using (2.25) and (2.26), we can see that

$$u_0=U(\eta,\tau),\quad u_1=\mathscr{R}\left\{\frac{\partial F}{\partial\eta}+\frac{\partial U}{\partial\eta}e^{-k\eta}\right\}e^{ik\psi}\quad (2.31)$$

and thus the dominant steady streaming is given by

$$u_1^{(s)}=\mathscr{R}\frac{e^{ik\psi}}{2\pi}\int_0^{2\pi}\frac{\partial F}{\partial\eta}d\tau.\quad (2.32)$$

3. The steady streaming when $kR \ll 1$

When $kR \ll 1$, so that the wavelength of the wall is very much greater than the amplitude of oscillation of a fluid particle far from the wall, we look for a solution

$$F(\eta,\tau) = F_0(\eta,\tau) + ikR\,F_1(\eta,\tau) + \dots. \tag{3.1}$$

When this is substituted into (2.27) and like powers of ikR are equated, we obtain the following equation for F_0:

$$\left[2\frac{\partial}{\partial\tau} - \left(\frac{\partial^2}{\partial\eta^2} - k^2\right)\right]\left(\frac{\partial^2}{\partial\eta^2} - k^2\right)F_0 = 0. \tag{3.2}$$

In addition F_0 must satisfy the following boundary conditions deduced from (2.28):

$$\left.\begin{array}{l} F_0 = 0,\ F_0' = -2^{\frac{1}{2}}\cos(\tau + \tfrac{1}{4}\pi) \quad \text{on} \quad \eta = 0, \\ \left.\begin{array}{l} F_0' \to 0 \\ F_0 \to 0 \end{array}\right\} \quad \text{as} \quad \eta \to \infty. \end{array}\right\} \tag{3.3}$$

Because we are admitting only those solutions which are periodic in τ, we easily find that

$$F_0 = 2^{-\frac{1}{2}}\left\{\frac{e^{i\tau+\frac{1}{4}i\pi}}{\sigma - k}[e^{-\sigma\eta} - e^{-k\eta}] + \frac{e^{-i\tau-\frac{1}{4}i\pi}}{\bar{\sigma} - k}[e^{-\bar{\sigma}\eta} - e^{-k\eta}]\right\}, \tag{3.4}$$

where $\sigma^2 = k^2 + 2i$, $|\arg\sigma| < \frac{1}{2}\pi$, and an overbar denotes 'complex conjugate'.

The equation of $O(ikR)$ obtained when (3.1) is substituted into (2.27) is

$$\left[2\frac{\partial}{\partial\tau} - \left(\frac{\partial^2}{\partial\eta^2} - k^2\right)\right]\left(\frac{\partial^2}{\partial\eta^2} - k^2\right)F_1 = -U(F_0'' - k^2F_0) + U''F_0, \tag{3.5}$$

with boundary conditions

$$\left.\begin{array}{l} F_1 = F_1' = 0 \quad \text{on} \quad \eta = 0, \\ \left.\begin{array}{l} F_1' \to 0 \\ F_1 \to 0 \end{array}\right\} \quad \text{as} \quad \eta \to \infty. \end{array}\right\} \tag{3.6}$$

Substituting (3.4) into the right-hand side of (3.5) we find that F_1 can be written as the sum of two terms, one of which is proportional to $e^{2i\tau}$ and has zero time average. This term gives no contribution to the steady streaming, and therefore attention is focused on the term independent of τ which we denote by $F_1^{(s)}$.

If we write

$$F_1^{(s)}(\eta) = f_1^{(s)}(\eta) + \bar{f}_1^{(s)}(\eta), \tag{3.7}$$

then we find

$$\left[\frac{d^2}{d\eta^2} - k^2\right]^2 f_1^{(s)} = \frac{2^{\frac{1}{2}}}{4}\left\{(\sigma + k)\,(1 - e^{-\eta(1-i)})\,e^{-\sigma\eta+\frac{1}{4}i\pi} - \frac{2i}{\sigma - k}(e^{-\sigma\eta} - e^{-k\eta})\,e^{-\eta(1-i)+\frac{1}{4}i\pi}\right\}. \tag{3.8}$$

The solution of this is

$$f_1^{(s)} = A\,e^{-\sigma\eta} + B\,e^{-\sigma\eta-\eta(1-i)} + C\,e^{-k\eta-\eta(1-i)} + D\,e^{-k\eta} + E\eta\,e^{-k\eta}, \tag{3.9}$$

where the coefficients may easily be determined from (3.8) and the boundary conditions (3.6).

It is more instructive to consider the form of (3.9) when k is very large or very small. We first consider the case when $k \gg 1$, i.e. the wavelength of the wall is much smaller than the thickness of the Stokes layer. It can be seen that (3.9) is exponentially small unless $\eta \leqslant O(1/k)$ and so we define a new scaled variable $\eta' = k\eta$. This implies that the steady streaming associated with $f_1^{(s)}$ is confined to a boundary layer whose thickness is of the order of a wavelength, and this is much smaller than the thickness of the Stokes layer. We find

$$f_1^{(s)} \sim (1/24k^4)(3\eta'^2 + \eta'^3)\,e^{-\eta'} + O(1/k^5) \tag{3.10}$$

and hence, from (2.32), that the dominant steady streaming is

$$u_1^{(s)} \sim -(R/12k^2)\,\eta'(6-\eta'^2)\,e^{-\eta'}\sin k\psi + O(R/k^3). \tag{3.11}$$

This steady streaming is sketched in figure 1.

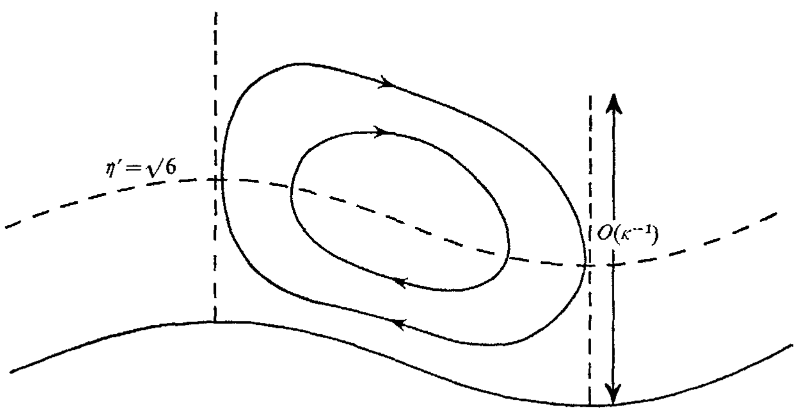

FIGURE 1. Sketch of steady streaming when $kR \ll 1$ and $k \gg 1$.

Let us now consider the case $k \ll 1$, i.e. the wavelength of the wall is much larger than the thickness of the Stokes layer. We see from (3.9) that $f_1^{(s)}$ now consists of two parts. One of these decays to zero in a length scale of the order of the thickness of the Stokes layer ($\eta \to \infty$), whilst the other decays over a much larger length scale of the order of a wavelength ($\eta' \to \infty$). Thus we may expand $f_1^{(s)}$ in powers of k in two regions: one where $\eta \sim O(1)$ and the other where $\eta' \sim O(1)$.

Thus when $\eta \sim O(1)$, the solution in the Stokes layer or inner region is

$$f_1^{(s)} \sim k\{-\tfrac{1}{8}\eta\,e^{-\eta}\sin\eta - \tfrac{3}{8}e^{-\eta}\cos\eta - \tfrac{1}{4}e^{-\eta}\sin\eta - \tfrac{1}{32}e^{-2\eta} + \tfrac{13}{32} - \tfrac{3}{16}\eta\} + O(k^2) \tag{3.12}$$

and we therefore find that

$$u_1^{(s)} \sim \tfrac{1}{2}k^2R\{\tfrac{1}{2}\eta\,e^{-\eta}\cos\eta - \tfrac{1}{2}\eta\,e^{-\eta}\sin\eta - \tfrac{1}{2}e^{-\eta}\cos\eta - 2\,e^{-\eta}\sin\eta - \tfrac{1}{4}e^{-2\eta} + \tfrac{3}{4}\}\sin k\psi + O(k^3R). \tag{3.13}$$

It can be shown that this is identical to the steady streaming predicted by the theory of Schlichting (1932) for oscillating flow over a curved boundary. This implies that, if $k \ll 1$ and $kR \ll 1$, no restriction need be placed on the amplitude of the wave a for Schlichting's theory to hold.

When $\eta' \sim O(1)$ then we find that $f_1^{(s)}$ may be written as

$$f_1^{(s)} \sim -\tfrac{3}{16}\eta' e^{-\eta'} + O(k) \tag{3.14}$$

and then
$$u_1^{(s)} \sim \tfrac{3}{8}k^2R(1-\eta')e^{-\eta'}\sin k\psi + O(k^3R). \tag{3.15}$$

Therefore in this region the steady streaming generated within the Stokes layer decays to zero. The solution (3.15) was found by Schlichting (1932) when solving for flow in the outer region for small values of his steady streaming Reynolds number, but it is due originally to Rayleigh (1884) who studied an analogous problem. In both cases the stream function from which (3.15) is derived satisfies the biharmonic equation with η' and ψ as independent variables.

The steady streaming predicted by (3.13) and (3.15) is sketched in figure 2.

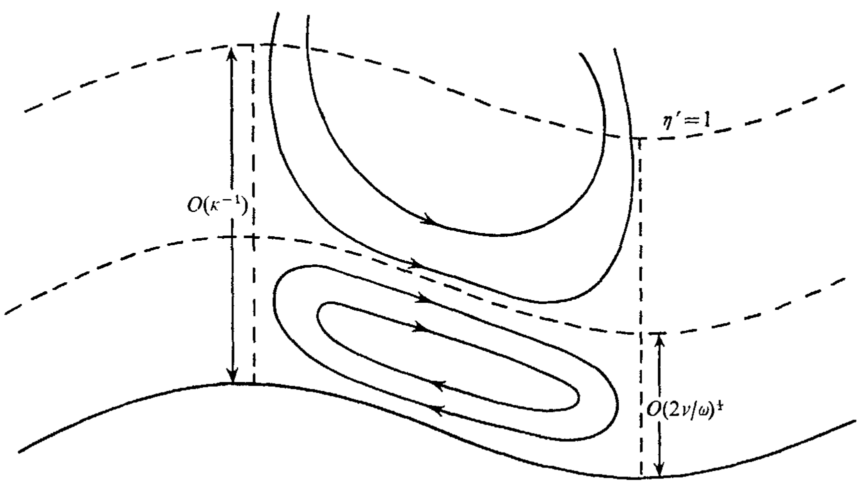

FIGURE 2. Sketch of steady streaming when $kR \ll 1$ and $k \ll 1$.

4. The solution when $kR \gg 1$

When kR is large, so that the amplitude of a fluid particle oscillation far from the wall is much greater than the wavelength of the wall, we may expect from (2.27) that the governing equation for the flow, away from any viscous boundary layers, is

$$U(F'' - k^2F) - U''F = 0; \tag{4.1}$$

we will refer to this as the inviscid equation and its solutions as inviscid solutions. In stability theory equation (4.1) is often referred to as the Rayleigh equation. We may note that, unlike the similar situation in stability theory, the time variable τ appears only as a parameter in (4.1). This is because of our insistence on periodic solutions which implies that $\partial/\partial\tau \sim O(1)$ and hence $(2/ikR)\,\partial/\partial\tau \ll 1$ in (2.27). This parametric property of τ is important, for it means that we may make extensive use of the theory of Benjamin (1959) for the steady problem.

As in stability theory (see, for example, Stuart 1963) equation (4.1) has a singular point at any position $\eta = \eta_c$ where $U = 0$ and consequently its solutions cease to be approximate solutions to the full equation (2.27) even when kR is very large. In fact we find by the method of Frobenius that the formal expansion of one of the solutions to (4.1) involves a term in $(\eta-\eta_c)\log(\eta-\eta_c)$ and so the correct form of the approximate solution is in doubt until the appropriate branch of the logarithm is decided. This ambiguity is resolved from a consideration of the full equation (2.27) in the vicinity of the critical point $\eta = \eta_c$, this necessarily takes into account the effects of viscosity. Tollmien (1929) first demonstrated that if the logarithm is expressed as $\log(\eta-\eta_c)$ when $\eta > \eta_c$, then it is to be replaced by $\log(\eta_c-\eta)-i\pi$ when $\eta < \eta_c$ providing $U'_c > 0$ (a suffix c denotes 'evaluated at $\eta = \eta_c$'). If $U'_c < 0$, then the logarithm is to be replaced by $\log(\eta_c-\eta)+i\pi$ when $\eta < \eta_c$.

In order to solve (4.1) we make the further assumption that $k \ll 1$, so that the wavelength of the wall is much greater than the thickness of the Stokes layer. Then, following Benjamin (1959), we find that the solution to (4.1), which is uniformly valid in η and satisfies the boundary condition (2.28) at infinity, is

$$F = A(\tau)\,U\,e^{-k\eta}\left\{1+k\int_\eta^\infty\left[\left(\frac{U_\infty}{U}\right)^2-1\right]d\eta+O(k^2)\right\}, \tag{4.2}$$

where U_∞ is the limit of U as $\eta\to\infty$ ($U_\infty = \cos\tau$). This solution is due originally to Lighthill (1957). Although the integral in (4.2) is generally a second-order term, near a critical point $\eta = \eta_c$ it becomes dominant and at $\eta = \eta_c$ exactly the integral is infinite. However, as $\eta\to\eta_c$ the zero in U cancels the singularity in the integral, and the whole expression gives the finite value

$$F_c = Ak(U_\infty^2/U'_c)\,e^{-k\eta_c} \tag{4.3}$$

if $U'_c \neq 0$. If in (4.2) there exist $\eta_{c_i}, i = 1, 2, \ldots, n$, such that $U_{c_i} = 0$ and

$$\eta < \eta_{c_1} < \eta_{c_2} < \ldots < \eta_{c_n},$$

then to obtain a more explicit form of (4.2), we indent the path of integration by circuiting each singularity by a small semicircle, under the real axis if $U'_{c_i} > 0$, above if $U'_{c_i} < 0$. We find that (4.2) now becomes

$$F = A\,U\,e^{-k\eta}\left\{1-i\pi k\sum_{i=1}^{n}\left(\frac{U_\infty}{U_{c_i}}\right)^2\cdot\frac{U''_{c_i}}{|U'_{c_i}|}+k\mathscr{P}\int_\eta^\infty\left[\left(\frac{U_\infty}{U}\right)^2-1\right]d\eta+O(k^2)\right\}, \tag{4.4}$$

where $\mathscr{P}$ denotes the 'principal value' of the integral in the sense of Hadamard (1923). This principal value is defined clearly by Mangler (1952) in his detailed analysis of certain types of integrals which arise in theoretical aerodynamics. The choice of contour is made so that the appropriate branch of the logarithm is chosen correctly on either side of $\eta = \eta_{c_i}$. If there exists a critical point $\eta = \eta_{c_m}$ where both U_{c_m} and U'_{c_m} are zero (but from (2.25) $U''_{c_m} \neq 0$), then we find that (4.4) must now include a term

$$\pm A\,U\,e^{-k\eta}i\pi k\frac{2}{3}\left(\frac{U_\infty}{U''_{c_m}}\right)^2\left[\frac{U''_{c_m}U^{\text{iv}}_{c_m}}{U''^2_{c_m}}-\frac{8}{9}\left(\frac{U'''_{c_m}}{U''_{c_m}}\right)^3\right], \tag{4.5}$$

where the plus sign is taken if the contour is indented below the singularity, the minus sign if indented above. In order to decide which of the contours to take, we would need to consider the solution of the full equation (2.27) near $\eta = \eta_{c_m}$. This is not examined here as we shall not, in fact, require the information. We note that F is singular at $\eta = \eta_{c_m}$ as the differential equation (4.1) implies.

Near $\eta = \eta_c$, and keeping τ constant, it is possible to expand (4.4) in the following Taylor series,

$$F \sim A[(\eta-\eta_c)\,U'_c+\tfrac{1}{2}(\eta-\eta_c)^2\,U''_c+O(\eta-\eta_c)^3]\,e^{-k\eta_c} \\ +Ak(U^2_\infty/U'_c)\{1+(U''_c/U'_c)\,(\eta-\eta_c)\log(\eta-\eta_c)+C(\eta-\eta_c) \\ +O[(\eta-\eta_c)^2\log(\eta-\eta_c)]\}\,e^{-k\eta_c}+O(k^2), \tag{4.6}$$

where C depends on τ; the evaluation of C depends on the behaviour of the integrand in (4·4) over the whole of the range of integration and not just locally as with the other terms. It is not evaluated here and is included solely to demonstrate the procedure whereby (4.6) is matched onto a solution valid at the critical point $\eta = \eta_c$. As mentioned before, when $\eta < \eta_c$, $\log(\eta-\eta_c)$ is replaced by $\log(\eta_c-\eta)-i\pi$ if $U'_c > 0$, and by $\log(\eta_c-\eta)+i\pi$ if $U'_c < 0$.

Following Reid (1965), we introduce the small parameter

$$\left.\begin{aligned} \epsilon &= [ikRU'_c]^{-\frac{1}{3}}, \\ \arg\epsilon &= -\tfrac{1}{6}\pi \quad (U'_c > 0), \\ \arg\epsilon &= \tfrac{5}{6}\pi \quad (U'_c < 0), \end{aligned}\right\} \tag{4.7}$$

and introduce the new scaled variables for the neighbourhood of $\eta = \eta_c$

$$\lambda = (\eta-\eta_c)\,\epsilon^{-1}, \quad G = F\epsilon^{-1}. \tag{4.8}$$

When these are substituted into (2.27) the singularity which is present in (4.1) no longer exists, for now the highest derivative is not lost but is of the same order as a retained inertial term (see (4.10)). In addition, we may notice that τ again appears only as a parameter, this being essential to the subsequent analysis, for, in deriving equations (4.10) to (4.12), we expand U as a Taylor series around $\eta = \eta_c$ keeping τ fixed. Thus the viscous effects associated with the critical points η_c are confined to thin layers of thickness $O[\epsilon(2\nu/\omega)^{\frac{1}{2}}]$.

We expand G in the following manner

$$G = G_{00}+(\epsilon\log|\epsilon|)\,G_{10}+\epsilon G_{11}+O(\epsilon\log|\epsilon|)^2, \tag{4.9}$$

where the G_{ij} are functions of λ and τ. If we substitute (4.9) into (2.27) and expand U in a Taylor series around $\eta = \eta_c$ keeping τ fixed, then we obtain the following equations for the G_{ij} on equating like powers of ϵ, etc.

$$\left(\frac{\partial^2}{\partial\lambda^2}-\lambda\right)\frac{\partial^2 G_{00}}{\partial\lambda^2} = 0, \tag{4.10}$$

$$\left(\frac{\partial^2}{\partial\lambda^2}-\lambda\right)\frac{\partial^2 G_{10}}{\partial\lambda^2} = 0, \tag{4.11}$$

$$\left(\frac{\partial^2}{\partial\lambda^2}-\lambda\right)\frac{\partial^2 G_{11}}{\partial\lambda^2} = \frac{U''_c}{2U'_c}\left(\lambda^2\frac{\partial^2 G_{00}}{\partial\lambda^2}-2G_{00}\right). \tag{4.12}$$

We will now focus attention onto the viscous layer formed on the wall, i.e. $\eta_c = 0$. This will enable us to determine the function $A(\tau)$ in (4.2) and (4.4). Therefore, in what follows, wherever a suffix c would have occurred we now use a suffix 0 to emphasize that we are considering this layer. We may deduce the boundary conditions to be imposed on equations (4.10) to (4.12) from (2.28), and these are

$$\left.\begin{aligned} G_{00} = 0, \quad \partial G_{00}/\partial\lambda = -U_0' \\ G_{1j} = \partial G_{1j}/\partial\lambda = 0 \quad (j = 0, 1), \end{aligned}\right\} \text{ on } \lambda = 0, \tag{4.13}$$

together with the requirement that G should match onto the inviscid solution F, assuming a common region of validity.

Following Reid (1965), we may write the general solutions to (4.10) and (4.11) as

$$G_{i0} = a_{i0} + b_{i0}\lambda + c_{i0}\int_{\infty_1}^{\lambda} d\lambda \int_{\infty_1}^{\lambda} \mathrm{Ai}(\lambda)\, d\lambda + d_{i0}\int_{\infty_2}^{\lambda} d\lambda \int_{\infty_2}^{\lambda} \mathrm{Ai}(\lambda e^{\frac{2}{3}i\pi})\, d\lambda \quad (i = 0, 1), \tag{4.14}$$

where the a_{i0}, etc. are functions of τ which are chosen to satisfy the boundary and matching conditions. The function $\mathrm{Ai}(\lambda)$ is the well-known Airy function and a property of this is that it is exponentially small at ∞ for $|\arg\lambda| < \frac{1}{3}\pi$, this being the sector in which ∞_1 lies. It is exponentially large elsewhere except on the lines $|\arg\lambda| = \frac{1}{3}\pi$ and $\arg\lambda = -\pi$. Thus $Ai(\lambda e^{\frac{2}{3}i\pi})$ is exponentially small for $-\pi < \arg\lambda < -\frac{1}{3}\pi$, this being the sector in which ∞_2 lies. Thus we see from (4.7) and (4.8) that when $U_0' > 0$ we must insist that $d_{i0} \equiv 0$ in order that the solution may not be exponentially large at infinity and therefore impossible to match with the inviscid solution. In the same way when $U_0' < 0$ we must insist that $c_{i0} \equiv 0$.

We may further deduce from the well-known properties of the Airy function that when $U_0' > 0$, the boundary conditions (4.13) are satisfied by G_{00} and G_{10} if

$$\left.\begin{aligned} \frac{b_{00} + U_0'}{a_{00}} = \frac{-2\pi}{3^{\frac{7}{6}}\Gamma(\frac{2}{3})}, \\ \frac{b_{10}}{a_{10}} = \frac{-2\pi}{3^{\frac{7}{6}}\Gamma(\frac{2}{3})}, \end{aligned}\right\} \tag{4.15}$$

where $\Gamma(x)$ is the gamma function, cf. Benjamin (1959). If $U_0' < 0$, then the right-hand sides of the expressions in (4.15) must each contain a factor $e^{\frac{2}{3}i\pi}$.

The general solution to (4.12) can be written as

$$G_{11} = \frac{U_0''}{2U_0'}[2a_{00}e^{\frac{1}{6}i\pi}N(\lambda e^{-\frac{1}{6}i\pi}) + b_{00}\lambda^2] + a_{11} + b_{11} + \text{exponentially decaying terms}, \tag{4.16}$$

where $N(x)$ is just the function referred to by Stuart (1963), who reproduces a table of its values due to Holstein (1950). This function is regular at the origin and has the following behaviour as $|\lambda| \to \infty$:

$$\left.\begin{aligned} N(\lambda e^{-\frac{1}{6}i\pi}) \sim |\lambda|\log(|\lambda|), \quad \arg\lambda = \tfrac{1}{6}\pi; \\ N(\lambda e^{-\frac{1}{6}i\pi}) \sim -|\lambda|\log(|\lambda|) + \pi i|\lambda|, \quad \arg\lambda = -\tfrac{5}{6}\pi. \end{aligned}\right\} \tag{4.17}$$

In order to make a meaningful match between these viscous solutions and the inviscid solution (4.4), we need to specify the size of k more carefully. In fact we assume here that

$$k = \epsilon k', \tag{4.18}$$

where $|k'| \sim O(1)$, but other scalings of k could be treated in a similar way to the following. Thus (4.4) gives the solution to the inviscid equation correct to $o(\epsilon^2)$, and we may also note that the full solution to (4.1) would give the solution to (2.27) correct to $o(\epsilon^3)$.

We therefore expand $A(\tau)$ in (4.4) as

$$A(\tau) = A_{00}(\tau) + (\epsilon \log|\epsilon|)\, A_{10}(\tau) + \epsilon A_{11}(\tau) + O(\epsilon \log|\epsilon|)^2 \tag{4.19}$$

and hence, writing (4.6) in terms of the viscous-layer variables (4.8) and using (4.18), we have that as $\eta \to 0$ the inviscid solution behaves like

$$\begin{aligned} G \sim A_{00}\left\{k'\frac{U_\infty^2}{U_0'} + \lambda U_0'\right\} &+ \epsilon \log|\epsilon| \left\{A_{10}\left(k'\frac{U_\infty^2}{U_0'} + \lambda U_0'\right) + A_{00}k'\frac{U_\infty^2}{U_0'}\cdot\frac{U_0''}{U_0'}\lambda\right\} \\ &+ \epsilon\left\{A_{11}\left(k'\frac{U_\infty^2}{U_0'} + \lambda U_0'\right) + A_{00}\frac{\lambda^2}{2}U_0'' + A_{00}k'\frac{U_\infty^2}{U_0'}C\lambda + A_{00}k'\frac{U_\infty^2}{U_0'}\cdot\frac{U_0''}{U_0'}\lambda\log(|\lambda|)\right\} \\ &+ O(\epsilon\log|\epsilon|)^2. \end{aligned} \tag{4.20}$$

Therefore, matching term by term with the solution for G as $\lambda \to \infty$ we have

$$\left.\begin{aligned} a_{ij} &= A_{ij}k'U_\infty^2/U_0^2 \quad (i = 0, 1, j = 0, 1), \\ b_{00} &= A_{00}U_0', \\ b_{10} &= A_{10}U_0' + A_{00}k'(U_\infty^2/U_0')\,U_0''/U_0', \\ b_{11} &= A_{11}U_0' + A_{00}k'(U_\infty^2/U_0')\,C. \end{aligned}\right\} \tag{4.21}$$

In addition, it can be seen that the terms involving $\log|\lambda|$ and λ^2 in the $O(\epsilon)$ factor are automatically matched by the solution for G_{11} (4.16), when the property (4.17) is utilized. The matching could be continued in the same way to a higher order in ϵ if desired. We may now deduce from (4.15) and (4.21) that

$$\left.\begin{aligned} A_{00} &= \left[\pm\frac{2\pi k(kR)^{\frac{1}{3}}}{3^{\frac{7}{6}}\Gamma(\frac{2}{3})}\cdot\frac{U_\infty^{\frac{2}{3}}\,e^{\pm\frac{1}{6}i\pi}}{(U_0')^{\frac{5}{3}}}\right]^{-1}, \\ A_{10} &= k(kR)^{\frac{1}{3}}\frac{U_\infty^2 U_0''}{(U_0')^{\frac{8}{3}}}A_{00}^2, \end{aligned}\right\} \tag{4.22}$$

where the plus sign is taken if $U_0' > 0$, and the minus sign is taken if $U_0' < 0$.

We now consider what happens in the viscous layer near a point $\tau = \tau_0$ when $U_0' = 0$. Near such a point

$$U \sim \eta(\tau-\tau_0)\,[\partial U_0'/\partial\tau]_{\tau=\tau_0} + \tfrac{1}{2}\eta^2[U_0'']_{\tau=\tau_0} + \cdots \tag{4.23}$$

and, in order to create a balance between the inertial and viscous terms in (2.27), we are led to the following scalings on $\tau - \tau_0$ and η

$$\gamma = \eta\delta^{-1}, \quad H = F\delta^{-1}. \quad T = (\tau-\tau_0)\,\delta^{-1}, \tag{4.24}$$

where δ is a small parameter defined by

$$\left.\begin{aligned} \delta &= \{ikR[U_0'']_{\tau=\tau_0}\}^{-\frac{1}{4}}, \\ \arg\delta &= -\tfrac{1}{8}\pi, \quad [U_0'']_{\tau=\tau_0} > 0, \\ \arg\delta &= +\tfrac{1}{8}\pi, \quad [U_0'']_{\tau=\tau_0} < 0. \end{aligned}\right\} \tag{4.25}$$

Substituting (4.25), (4.24) and (4.23) into (2.27) the equation becomes

$$\frac{\partial^4 H}{\partial\gamma^4} + (T\gamma - \tfrac{1}{2}\gamma^2)\frac{\partial^2 H}{\partial\gamma^2} + H = O(\delta), \tag{4.26}$$

noting that from our definition of U (2.25)

$$\left[\frac{\partial U_0'}{\partial \tau}\Big/U_0''\right]_{\tau=\tau_0} = -1. \tag{4.27}$$

In addition, the boundary conditions to be satisfied on the wall become, from (2.28)

$$H = 0, \quad \frac{\partial H}{\partial \gamma} = -\delta T\left[\frac{\partial U_0'}{\partial \tau}\right]_{\tau=\tau_0} + O(\delta^2). \tag{4.28}$$

The first-order equation for H is obtained by putting the right-hand side of (4.26) equal to zero and we again note the important feature that the time variable T occurs only as a parameter. Four independent solutions to this equation can be found in the form of contour integrals and are described in detail by Lyne (1970). However, these integrals can only be evaluated asymptotically and, for our purposes, hold no advantage over the following simpler analysis. We observe that one solution of (4.26) is

$$H_1 = h(T)\,[T\gamma - \tfrac{1}{2}\gamma^2]. \tag{4.29}$$

We may infer from (4.26) that the other three solutions are regular at $\gamma = 0$, and, by assuming the following behaviour of H as $|\gamma| \to \infty$,

$$H \sim \gamma^{\sigma} e^{h_1\gamma + h_2\gamma^2}\{a_0 + a_1/\gamma + \ldots\}, \tag{4.30}$$

we may show that they have the following asymptotic forms

$$H_2 \sim \gamma^{-1}\{b_0 + b_1/\gamma + \ldots\}, \tag{4.31}$$

$$H_3 \sim \gamma^{-\frac{1}{2}(5 - T^2/\sqrt{2})} e^{+(T\gamma - \frac{1}{2}\gamma^2)/\sqrt{2}}\{c_0 + c_1/\gamma + \ldots\}, \tag{4.32}$$

$$H_4 \sim \gamma^{-\frac{1}{2}(5 - T^2/\sqrt{2})} e^{-(T\gamma - \frac{1}{2}\gamma^2)/\sqrt{2}}\{d_0 + d_1/\gamma + \ldots\}, \tag{4.33}$$

where the a's etc., are functions of T.

Because $\eta > 0$ we see from (4.24) and (4.25) that, if $T \sim O(1)$, then $d_i \equiv 0$ in order that H may not be exponentially large and therefore impossible to match with the inviscid solution. That the function H does match naturally onto the inviscid solution F may be seen by expanding (4.4) with (4.5) at $\tau = \tau_0$ for small η

$$F \sim A\left\{\frac{2kU_\infty^2}{3U_0''}\left[\frac{1}{\eta} - \frac{2}{3}\frac{U_0'''}{U_0''} + \left\{\frac{2}{3}\left(\frac{U_0'''}{U_0''}\right)^2 - \frac{5}{12}\frac{U_0^{\text{iv}}}{U_0''}\right\}\eta \right.\right.$$
$$\left.\left. + \left\{\frac{4}{9}\left(\frac{U_0'''}{U_0''}\right)^2 - \frac{1}{2}\frac{U_0'''U_0^{\text{iv}}}{U_0''^2}\right\}\eta^2\log\eta\right] + O(\eta^2) + O(k^2)\right\}. \tag{4.34}$$

Writing this in the scaled variables (4.24) we have when $\eta \sim 0$

$$H \sim O(Ak\delta^{-2}/\gamma). \tag{4.35}$$

The expression (4.18) implies that k is $O(\delta^{\frac{4}{3}})$ and, since the boundary conditions (4.28) imply that H is $O(\delta)$, we therefore require that

$$A = O(\delta^{\frac{5}{3}}), \tag{4.36}$$

in order that the inviscid solution should match onto H_2 in (4.31). However, when $U_0' \sim -\delta T U_0''$ (see (4.27)), we see from (4.22) that

$$A_{00} = O(\delta^{\frac{5}{3}}). \tag{4.37}$$

Thus the leading term in an expansion for A in terms of the small parameter δ near $\tau = \tau_0$ is of the same order of magnitude as that predicted by the leading term in an expansion in terms of ϵ elsewhere. This fact is of particular significance when we come to evaluate the steady streaming associated with this flow.

The expression (4.36) also implies that the function H_1 of (4.29) must be of $O(\delta^{\frac{8}{3}})$ in order that it may match onto the dominant term in η^2 of (4.34). (F is expanded at $\tau = \tau_0$ in (4.34) and so $T = 0$ in the expression for H_1.) Thus H_1 is not present in the first-order solution. The two remaining functions H_2 (4.31) and H_3 (4.32) (H_4 was dismissed because it was exponentially large at infinity) can now be made to satisfy the boundary conditions (4.28), and this gives rise to the value of the leading term in an expansion for A.

We may also see how this first-order solution for H matches onto the viscous solution G_{00} as $|T| \to \infty$. If we use the W.K.B. method and assume that for $\arg\gamma = -\frac{1}{8}\pi$ ($U_0'' < 0$) and $\arg T = -\frac{1}{8}\pi$ ($\tau > \tau_0$)

$$H \sim g(T)\, e^{\sqrt{(T)}\,\phi(\gamma)}\{f_0(\gamma) + (1/\sqrt{T})f_1(\gamma) + \ldots\}, \tag{4.38}$$

then we find that

$$H \sim g_1 + g_2\gamma + g_3\gamma^{-\frac{5}{4}}\exp\left(-\tfrac{2}{3}|T\gamma^{\frac{3}{2}}|\,e^{\frac{1}{4}i\pi}\right) + g_4\gamma^{-\frac{5}{4}}\exp\left(+\tfrac{2}{3}|T\gamma^{\frac{3}{2}}|\,e^{\frac{1}{4}i\pi}\right), \tag{4.39}$$

where the g_i are functions of T. We may see immediately that $g_4 \equiv 0$ otherwise H would be exponentially large. If we write the expression for G_{00} (4.14) in terms of the scaled variables (4.24), then we find that, for $U_0' \sim -\delta T U_0'' > 0$

$$H \sim O(\delta) + O(\delta^{\frac{8}{3}})\gamma + O(\delta)\int_{\infty_1}^{|\gamma T^{\frac{1}{2}}|\,e^{\frac{1}{4}i\pi}} d\lambda \int_{\infty_1}^{\lambda} \mathrm{Ai}\,(\lambda)\,d\lambda \tag{4.40}$$

and we may show that for large T the double integral gives rise to the following asymptotic representation

$$H \sim O(\delta) + O(\delta^{\frac{8}{3}})\gamma + O(\delta)\gamma^{-\frac{5}{4}}\exp\left(-\tfrac{2}{3}|T\gamma^{\frac{3}{2}}|\,e^{\frac{1}{4}i\pi}\right) \tag{4.41}$$

(see Reid 1965).

Thus as well as ensuring a match with (4.39) we see that it is also consistent with H being $O(\delta)$. In addition the $O(\delta^{\frac{8}{3}})\gamma$ term again demonstrates the order of magnitude of H_1 with which it must match, and this agrees with the order of magnitude found previously. We may perform similar analyses when either $U_0'' > 0$ or $\tau < \tau_0$ (or both) and we find again that the functions match onto each other consistently.

Thus this first-order solution for H, though not found explicitly, satisfies all our requirements: it satisfies the boundary conditions on the wall and matches onto both the inviscid solution F and the viscous solution G_{00}. It also enables us to find the leading term in an expansion for the function A in (4.4) when $\tau \sim \tau_0$ and, as mentioned before, the important feature of this is that its order of magnitude is the same as that predicted by the leading term in an expansion elsewhere.

The physical significance of this region of thickness $O[\delta(2\nu/\omega)^{\frac{1}{2}}]$ near $\tau = \tau_0$ is that it represents the creation of another viscous layer of thickness $O[\epsilon(2\nu/\omega)]^{\frac{1}{2}}$. This breaks away from the viscous layer on the wall and propagates into the inviscid region moving with the point $\eta = \eta_{c_1}(\tau)$ where $U_{c_1} = 0$. After a certain length of time a point is reached where U_{c_1} is again zero and now the viscous layer combines with another layer at $\eta = \eta_{c_2}$ and they both disappear. They reappear later as the two layers at $\eta = \eta_{c_2}$ and η_{c_3} respectively. It should be noted that during part of the period of oscillation there are no viscous layers away from the wall, whilst at other times there may be several. Indeed, when $U_\infty = 0$ there are an infinite number although their effect decays exponentially away from the wall.

We may easily see from matching with (4.6) that the viscous solutions G_{ij} given in (4.14) and (4.16) are immediately applicable for the solution of (4.10)–(4.12) in the viscous layers away from the wall. As we should expect, we find that the Airy function solutions are not required. If one had existed and decayed exponentially for $\eta > \eta_c$, then it would have increased exponentially for $\eta < \eta_c$ and this would have been intolerable. Similarly, the first-order solution H_2 (4.31) to the equation (4.26) is directly applicable in the regions where the viscous layers are either emerging or disappearing. As before, at the inception of such a region H_2 is matched to the first-order viscous solution G_{00}, but at its conclusion it now has to be matched onto the inviscid solution F. However, on closer investigation we see that these are now identical to first order in ϵ, and thus the matching is, in fact, unaffected. Because A_{00} is now $O(1)$ near such a region, we see from matching that H_2 must be $O(\delta^{-\frac{2}{3}})$ and this is $O(\delta^{-\frac{5}{3}})$ greater than in the equivalent region on the wall.

5. The steady streaming when $kR \gg 1$

We now concentrate on the evaluation of the steady streaming associated with this flow away from the viscous layer on the wall. From (2.32) we may see that if there were no viscous layers then the contribution of $O(a)$ would be given by

$$u_1^{(s)} = \mathscr{R}\frac{e^{ik\psi}}{2\pi}\int_0^{2\pi}\frac{\partial F}{\partial \eta}\,d\tau, \tag{5.1}$$

F being the inviscid solution and $\mathscr{R}$ denoting 'real part of'. However, as mentioned previously, near the viscous layer at a point $\eta = \eta_c$, the inviscid solution is identical to the viscous solution to first order in ϵ. Therefore, using (4.4) and (4.19), we find that the dominant contribution to the steady streaming is

$$u_1^{(s)} = \mathscr{R}\frac{e^{-k\eta+ik\psi}}{2\pi}\int_0^{2\pi}A_{00}\frac{\partial U}{\partial \eta}\,d\tau, \tag{5.2}$$

the neglected terms being of $O(kR)^{-\frac{1}{3}}$. Strictly, we do not know the value of the leading term for A in a region near the time τ_0 when $U_0' = 0$, but we inferred from (4.36) and (4.37) that it was the same order of magnitude as that predicted by A_{00}. More explicitly it is of $O(\delta^{\frac{5}{3}})$ and, because such a region exists for a time of $O(\delta)$, the error incurred from this source when using (5.2) is of $O(\delta^{\frac{8}{3}})$. This is $O(kR)^{-\frac{2}{3}})$ and is much smaller than the effect of the neglected terms. Should, however, the integration pass through a region where $U_c' \sim 0$ then the use of (5.2) must involve an error of $O(\delta H)$. We saw from the matching conditions outlined above that H is $O(\delta^{-\frac{2}{3}})$ and so the error is $O(\delta^{\frac{1}{3}})$. Because this is $O(kR)^{-\frac{1}{12}}$ it is very much larger than the error of $O(kR)^{-\frac{1}{3}}$ from the neglected terms. Nevertheless, when $kR \gg 1$, it is still vanishingly small compared to (5.2).

Using (2.25) and (4.22) we find that we can write (5.2) as

$$u_1^{(s)} = -\frac{2^{\frac{1}{2}}}{\pi}\Delta \sin\left(\tfrac{1}{6}\pi\right) e^{-(1+k)\eta} \sin k\psi$$
$$\times \left\{\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{\cos(t-\eta)\cos^{\frac{5}{3}} t(1+\sin 2t)}{\cos^{\frac{10}{3}} t + 2\Delta \cos^{\frac{5}{3}} t(1+\sin 2t)\cos\left(\tfrac{1}{6}\pi\right) + \Delta^2(1+\sin 2t)^2}\, dt\right\}, \quad (5.3)$$

where

$$\Delta = \frac{k(kR)^{\frac{1}{3}}}{2^{\frac{5}{6}} 3^{\frac{7}{6}}\, \Gamma\left(\frac{2}{3}\right)}. \quad (5.4)$$

More graphically

$$u_1^{(s)} = -k(kR)^{\frac{1}{3}} e^{-(1+k)\eta}\{I_1 \cos\eta + I_2 \sin\eta\} \sin k\psi, \quad (5.5)$$

where the integrals I_1 and I_2 may be evaluated numerically. These have been tabulated for a range of Δ by Lyne (1970).

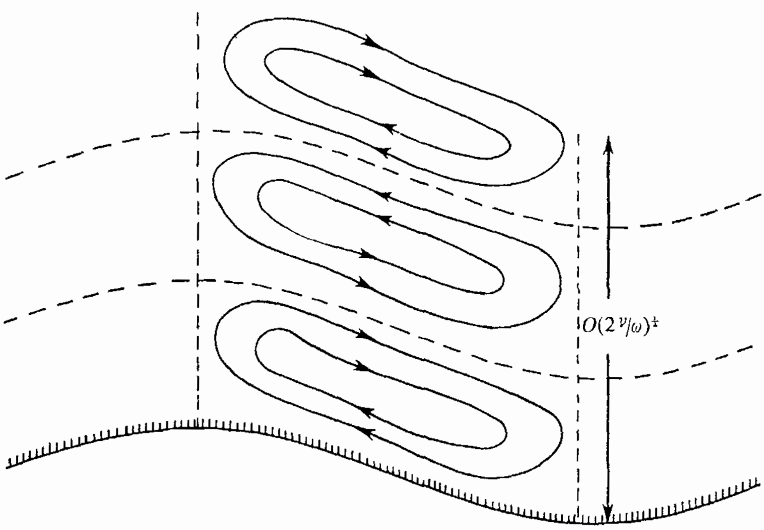

FIGURE 3. Sketch of steady streaming when $kR \gg 1$ and $k(kR)^{\frac{1}{3}} \sim O(1)$.
ɯɯ = viscous layer of thickness $O(kR)^{-\frac{1}{3}}\,(2\nu/\omega)^{\frac{1}{2}}$.

It is of some interest to evaluate (5.5) when $k(kR)^{\frac{1}{3}} \ll 1$, and after lengthy analyses given in detail by Lyne, we find

$$u_1^{(s)} = -\frac{\pi^{\frac{1}{2}}\,\Gamma\left(\frac{1}{6}\right) k(kR)^{\frac{1}{3}}}{2^{\frac{4}{3}} 3^{\frac{7}{6}} [\Gamma\left(\frac{2}{3}\right)]^2} \left[\cos\eta + \tfrac{3}{2}\sin\eta\right] e^{-(1+k)\eta} \sin k\psi + O[k(kR)^{\frac{1}{3}}]^{\frac{6}{5}} \quad (5.6)$$

and this gives

$$u_1^{(s)} = -0{\cdot}5927\, k(kR)^{\frac{1}{3}} \left[\cos\eta + \tfrac{3}{2}\sin\eta\right] e^{-(1+k)\eta} \sin k\psi + O[k(kR)^{\frac{1}{3}}]^{\frac{6}{5}}. \quad (5.7)$$

To summarize, the dominant steady streaming is of $O(a)$, and, if we assume $k \sim O(kR)^{-\frac{1}{3}}$ then, away from the viscous layer on the wall, it is given by (5.5) when $kR \gg 1$. The error is $O(kR)^{-\frac{1}{3}}$ almost everywhere, but, if η is within a distance $O(kR)^{-\frac{1}{4}}$ from a point where, at some time during a cycle, both U and U' are zero, then the error is of $O(kR)^{-\frac{1}{12}}$. This may be quite considerable even for very large values of kR. Should we then take $k(kR)^{\frac{1}{3}} \ll 1$, the steady streaming would be given by (5.7). Because the error is $O[k(kR)^{\frac{1}{3}}]^{\frac{1}{5}}$, this too may be quite large even for very small values of $k(kR)^{\frac{1}{3}}$. A sketch of the steady streaming predicted by (5.5) is given in figure 3 and shows a stacked structure of regions of recirculation the magnitudes of the circulations decreasing exponentially away from the wall.

6. Discussion

In all three cases considered the steady streaming is in the same direction immediately adjacent to the wall. When $kR \ll 1$, the streaming is confined to a layer of thickness $O(\kappa)^{-1}$ or the wavelength of the wall, and if this is much smaller than the thickness of the Stokes layer $(2\nu/\omega)^{\frac{1}{2}}$, only one region of recirculation exists (figure 1). When the wavelength is much greater than the Stokes layer thickness, two regions of recirculation exist (figure 2), that nearest the wall being confined to the Stokes layer. As pointed out previously, this steady streaming was first evaluated by Schlichting (1932).

When $kR \gg 1$ and $k(kR)^{\frac{1}{3}} \sim O(1)$, the steady streaming takes on a different character. Although confined within the Stokes layer, it now consists of several regions of recirculation of equal thickness (figure 3). This recirculatory flow is driven from within a thin viscous layer of thickness $O(kR)^{-\frac{1}{3}}(2\nu/\omega)^{\frac{1}{2}}$ formed on the wall, and this thickens to $O(kR)^{-\frac{1}{4}}(2\nu/\omega)^{\frac{1}{2}}$ when $U' \sim 0$. At some time during a period of oscillation there may exist one or more such viscous layers away from the wall. However, these layers adopt a passive role serving merely to eliminate the singularity arising from the inviscid equation (4.1). This is unlike the similar situation in stability theory and is a natural result of our insistence on periodic solutions.

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